Tut 1: Direction Field and a Recap of Sec. 2.1

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There are two parts in this tutorial note. In this first part, I will introduce the direction field. And in the second, I will recap the the contents of section 2.1 in the lecture notes. In this whole note, I use x as independent variable instead of t.

1 Direction Field

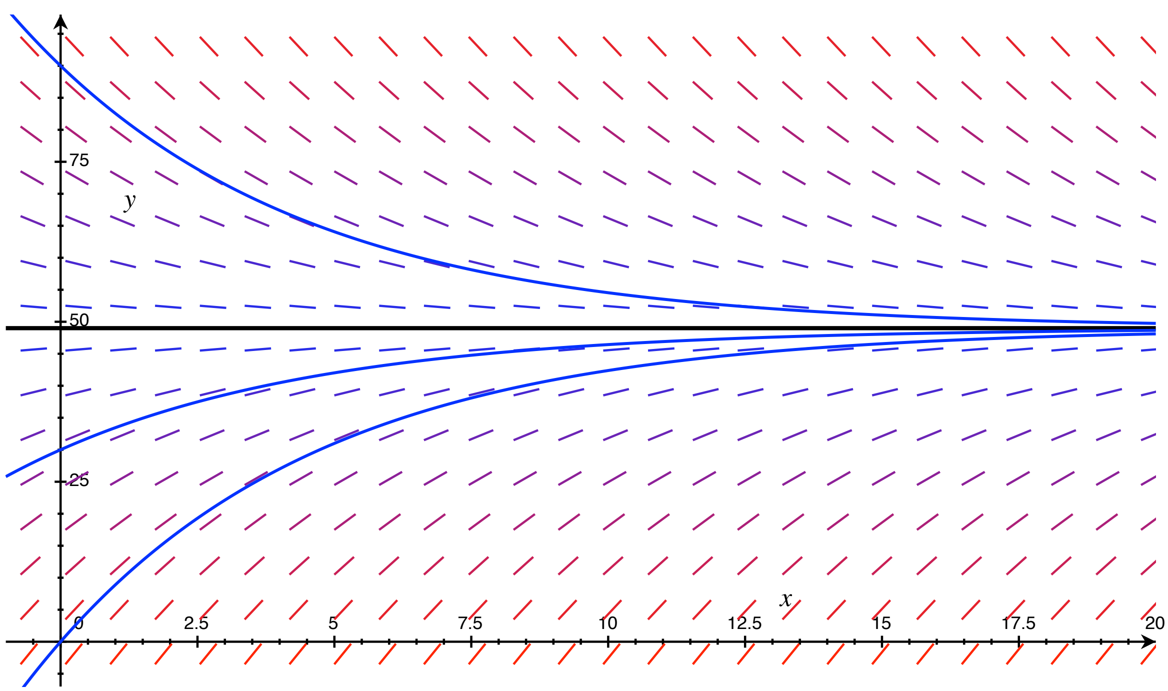
Direction fields are valuable tools in studying qualitative properties of solutions to first order ordinary differential equation (ODE)

$$\frac{dy}{dx} = F(x, y), \tag{1}$$

it is especially useful when you can not solve the equation. Now suppose you already have a solution curve for the ODE which runs through the point (x_0, y_0), then $F(x_0, y_0)$ should be the slope of the curve at (x_0, y_0) by the equation. This is key point of direction field. In the following, I will use specific examples to show you the meaning of direction fields.

Example 1. Motion of falling object:

$$\frac{dy}{dx} = g - \frac{\gamma y}{m} \qquad \text{with} \quad g = 9.8 \quad \text{and} \quad \gamma/m = 1/5. \tag{2}$$



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The above graph with these short lines is called a graph of the direction field to the differential equation (2). But why do we care about the direction field? Indeed, there are two reasons:

(a) Sketch of the solution curves. See these blue curve lines in the above picture.

(b) Long time behaviour.

Note that $y \equiv 49$ is a solution to (2) which is independent of variable x, henceforth we call such solution an **equilibrium solution**. By sketching the solution curves, we know that as x increase,

- if initial data $y_0 > 49$, it looks like all the solutions will decrease and approach to the equilibrium solution $y \equiv 49$.
- if initial data $y_0 < 49$, it looks like all the solutions will increase and approach to the equilibrium solution $y \equiv 49$.

Hence, no matter how initial data varies, the solution will always approach the equilibrium solution $y \equiv 49$, therefore we call it a **globally asymptotically stable solution**.

Example 2. Population dynamics:

$$\frac{dy}{dx} = ry(1 - y/K) \qquad \text{with} \quad r = 0.5. \tag{3}$$

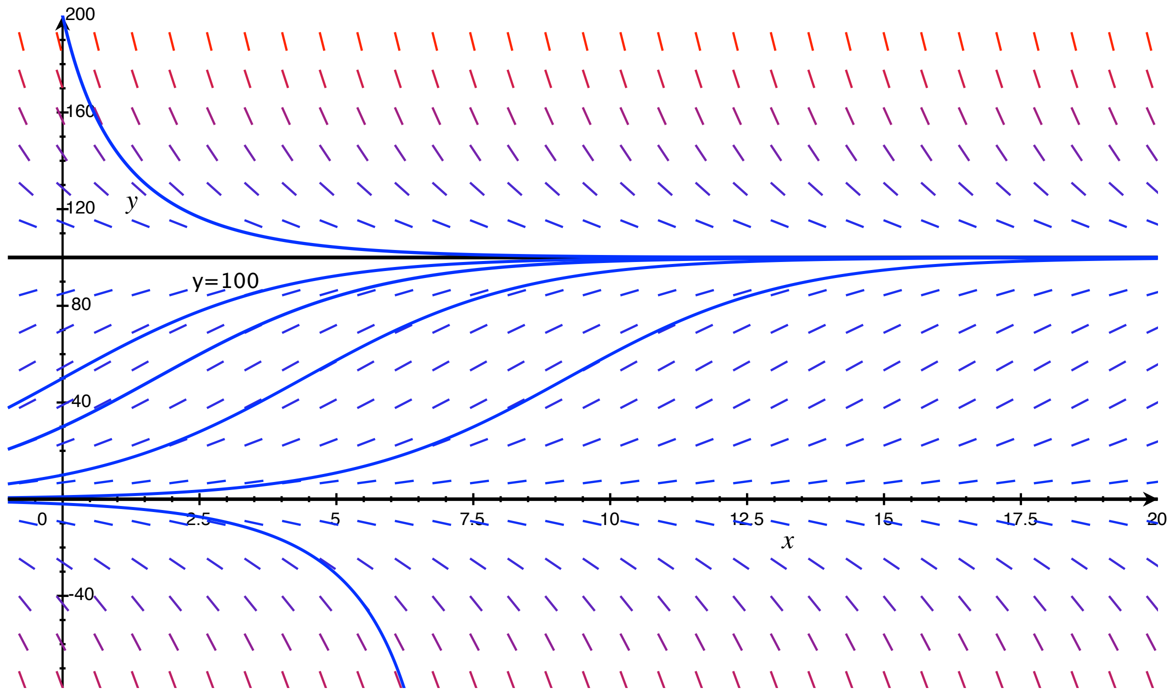
In this case, we have two equilibrium solutions:

$$y_1 \equiv 0 \qquad \text{and} \qquad y_2 \equiv K.$$

By scaling, we take

$$K = 100.$$

Then the graph of direction field is:



From this graph,

Table 1: Population dynamics

Initial data y_0	$y_0 > 100$	$y_0 = 100$	$0 < y_0 < 100$	$y_0 = 0$	$y_0 < 0$
as $x \to +\infty$	$y(x) \to 100$	$y(x) \equiv 100$	$y(x) \to 100$	$y(x) \equiv 0$	$y(x) \to -\infty$

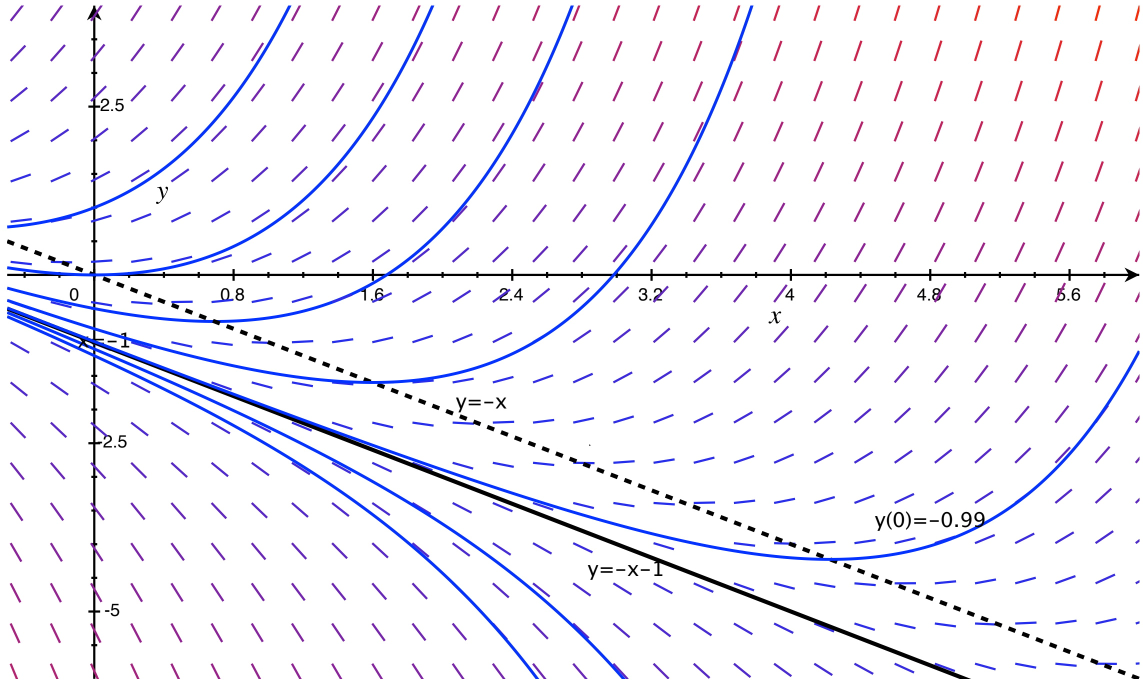
Thus as x goes to $+\infty$, for each $y_0 > 0$ the solution approaches the equilibrium solution $y_2 \equiv 100$. Therefore we say that the equilibrium solution $y_2 \equiv 100$ is an **asymptotically stable solution** of equation (3) (note that it is not globally stable). On the contrary, $y_1 \equiv 0$ is called an **unstable equilibrium solution** since the solution will go away from it even solution start very near zero.

In general, we call an equilibrium solution to a differential equation "asymptotically stable" if any solution start **near** the equilibrium solution approaches the equilibrium as x goes to infinity. Otherwise, it is called "unstable".

In the following example, we concern about a first order non-autonomous ODE.

Example 3.

$$\frac{dy}{dx} = y + x. \tag{4}$$



There is no equilibrium solution for this equation. As x goes to $+\infty$, the solution will diverge to $+\infty$ if the initial data is larger than -1, and diverge to $-\infty$ on the contrary.

2 Two Linear ODE Examples

The general form of an first order linear ODE is

$$\frac{dy}{dx} + p(x)y = q(x). \tag{5}$$

Example 1: If $p \equiv p_0$, $q \equiv q_0$ for some constants $p_0, q_0 \in \mathbb{R}$, then the linear ODE leads to an autonomous, linear ODE

$$\begin{cases} \dfrac{dy}{dx} + p_0 y = q_0, \\ \qquad y(0) = y_0. \end{cases} \tag{6}$$

Sol: We consider the equation by two cases.
Case 1: If $p_0 = 0$, then

$$\frac{dy}{dx} = q_0. \tag{7}$$

It is clear that

$$y = y_0 + q_0 x.$$

Case 2: If $p_0 \neq 0$, then

$$\frac{dy}{dx} = -p_0 y + q_0 = -p_0 \left(y - \frac{q_0}{p_0} \right).$$

If $y_0 = q_0/p_0$, then the solution is

$$y \equiv q_0/p_0.$$

Otherwise we admit $y(x) \neq q_0/p_0$ for any x (Try to prove it by contradiction argument). Hence

$$\frac{1}{y - q_0/p_0}\frac{dy}{dx} = -p_0, \tag{8}$$

which furthermore implies

$$\frac{d}{dx}\ln\left|y - \frac{q_0}{p_0}\right| = -p_0$$

by chain rule. Hence for some constant $c \in \mathbb{R}$, there holds that

$$\ln\left|y - \frac{q_0}{p_0}\right| = -p_0 x + c.$$

Taking exponential implies the general solution

$$y = \kappa e^{-px} + \frac{q}{p} \quad \text{with} \quad \kappa = e^c \text{ or } -e^c.$$

By the initial condition, one have

$$\kappa = y_0 - \frac{q_0}{p_0}, \quad \text{and} \quad y = \left(y_0 - \frac{q_0}{p_0}\right)e^{-p_0 x} + \frac{q_0}{p_0}.$$

In conclusion,

$$y = \begin{cases} y_0 + q_0 x, & \text{if} \quad p_0 = 0; \\ \left(y_0 - \frac{q_0}{p_0}\right)e^{-p_0 x} + \frac{q_0}{p_0}, & \text{if} \quad p_0 \neq 0. \end{cases} \tag{9}$$

Example 2: If $q \equiv 0$, then we consider

$$\begin{cases} \frac{dy}{dx} + p(x)y = 0, \\ \qquad y(0) = y_0. \end{cases} \tag{10}$$

Sol: If $y_0 = 0$, then $y \equiv 0$ is the solution. Otherwise rewriting the equation as

$$\frac{d}{dx}\ln|y| = \frac{1}{y}\frac{dy}{dx} = -p(x)$$

then the general solution can be represented as

$$y = \kappa \exp\left(-\int p(x)\,dx\right). \tag{11}$$

for some constants $\kappa \in \mathbb{R}$. By the initial condition,

$$y = y_0 \exp\left(-\int p(x)\,dx\right).$$